**Proof By Induction**

**Mathematical Induction** is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers (positive integers). It is done by proving that the first statement in the infinite sequence of statements is true, and then proving that if any one statement in the infinite sequence of statements is true, then so is the next one.

**Divisibility**

Prove by induction that 8 is a factor $7^{2n+1} + 1$ for $n \in N$

**Step 1:** Show true for $n = 1$

$7^{2(1)+1} + 1 = 7^3 + 1 = 344$

which is divisible by 8

**Step 2:** Assume true for $n = k$

$7^{2k+1} + 1$ is divisible by 8

**Step 3:** Prove true for $n = k + 1$

To prove: $7^{2(k+1)+1} + 1 = 7^{2k+3} + 1$ is divisible by 8

$f(k+1) - f(k) = 7^{2k+3} + 1 - (7^{2k+1} + 1) = 7^{2k}.7^3 + 1 - 7^{2k} + 7^1 - 1 = 7^{2k}(7^3 - 1) = 7^{2k}(336)$

which is divisible by 8

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in N$

The idea here is that if $f(k)$ and $f(k+1) - f(k)$ have common factors then $f(k+1)$ has this common factor also.

For example 6 is a factor of 12 and 36. Well 48 also has 6 as a factor.

If $f(k+1) - f(k)$ proves difficult try $f(k+1) + f(k)$ or the alternate method on the reverse of this page.

**Series**

Prove by induction that the sum of the first $n$ natural numbers $\sum n = \frac{n(n+1)}{2}$ for $n \in N$.

**Step 1:** Show true for $n = 1$

$1 = \frac{1(1+1)}{2}$

$1 = 1$

**Step 2:** Assume true for $n = k$

$1 + 2 + 3 + \cdots + k + 1 = \frac{k(k+1)}{2}$

**Step 3:** Prove true for $n = k + 1$

To prove:

$1 + 2 + 3 + \cdots + k + k + 1 = \frac{(k + 1)(k + 2)}{2}$

Proof:

$1 + 2 + 3 + \cdots + k + k + 1 = \frac{k(k+1)}{2} + k + 1$

Begin with the assumption

Add $k + 1$ to both sides and then work with RHS only.

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in N$

**Inequalities**

Prove by induction that $2^n \geq n^2$ for $n \geq 4$, $n \in N$

**Step 1:** Show true for $n = 4$

$2^4 \geq 4^2$

**Step 2:** Assume true for $n = k$

$2^k \geq k^2$

**Step 3:** Prove true for $n = k + 1$

To prove: $2^{k+1} \geq (k + 1)^2$

Since $2^k \geq k^2$

$2^k.2 \geq 2k^2$

Therefore $2^{k+1} \geq 2k^2$

So we need to show that $2k^2 \geq (k + 1)^2$

$2k^2 \geq k^2 + 2k + 1$

$k^2 - 2k - 1 \geq 0$

$k^2 - 2k + 1 - 2 \geq 0$

$(k - 1)^2 - 2 \geq 0$

Which is true for $k \geq 4$

$2^{k+1} \geq 2k^2 \geq (k + 1)^2$

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in N$

The above method is based on the idea that if $A > B$ and we can show that $B > C$ well then $A > C$
Divisibility (Alternate Method)
Prove by induction that 3 is a factor of $5^n - 2^n$ for all $n \in \mathbb{N}$ and $n \geq 1$.

**Step 1:** Show true for $n = 1$

$5^n - 2^n = 5^1 - 2^1 = 3$
which is divisible by 3

**Step 2:** Assume true for $n = k$

$5^k - 2^k$ is divisible by 3

**Step 3:** Prove true for $n = k + 1$

To prove: $5^{k+1} - 2^{k+1}$
is divisible by 3

Take the assumption and we know that it is a multiple of 3.

$5^k - 2^k = 3m$

$5^k = 3m + 2^k$

We can sub this into our proof.

$5^{k+1} - 2^{k+1}$

$= 5.5^k - 2.2^k$

$= 5.(3m + 2^k) - 2.2^k$

$= 15m + 5.2^k - 2.2^k$

$= 15m + 3.2^k$

$= 3(5m + 2^k)$
which is divisible by 3

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in \mathbb{N}$

Inequality ...Special Case

Prove by induction that $(1 + x)^n \geq 1 + nx$ for $x > -1$, $n \in \mathbb{N}$.

**Step 1:** Show true for $n = 1$

$(1 + x)^1 \geq 1 + x$

$1 + x \geq 1 + x$

**Step 2:** Assume true for $n = k$

$(1 + x)^k \geq 1 + kx$

**Step 3:** Prove true for $n = k + 1$

To prove:

$(1 + x)^{k+1} \geq 1 + (k + 1)x$

Proof:

Return to assumption

$(1 + x)^{k+1} \geq 1 + kx$

Multiply both sides by $1 + x$

$(1 + x)^{k+1}(1 + x) \geq (1 + kx)(1 + x)$

$(1 + x)^{k+1} \geq (1 + kx)(1 + x)$

We know the above is true so we try prove that

$(1 + kx)(1 + x) \geq 1 + (k + 1)x$

$1 + x + kx + kx^2 \geq 1 + kx + x$

$kx^2 \geq 0$

This is true for all values of $n$.

Therefore $(1 + x)^{k+1} \geq (1 + kx)(1 + x) > 1 + (k + 1)x$

So

$(1 + x)^{k+1} \geq 1 + (k + 1)x$

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in \mathbb{N}$

Inequality ...with Factorials

Prove, by induction, that $n! > 2^{n-1}$, for $n \geq 3$, $n \in \mathbb{N}$.

**Step 1:** Show true for $n = 3$

$n! > 2^{3-1}$

$3! > 2^2$

$6 > 4$

True

**Step 2:** Assume true for $n = k$

$k! > 2^{k-1}$

**Step 3:** Prove true for $n = k + 1$

$(k + 1)! > 2^{(k+1)-1}$

$(k + 1)! > 2^k$

We know the above is true so we try prove that

$(k + 1)2^{k-1} > 2^k$

$(k + 1)2^{k-1} > 2^k$

$2(k + 1) > 1$

$k + 1 > 2$

$k + 1 > 2$

$k > 1$

This is true for all values $n \geq 3$

Therefore $(k + 1)! > (k + 1)2^{k-1} > 2^k$

So

$(k + 1)! > 2^k$

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore by induction it is true for all $n \in \mathbb{N}$