ALGEBRA

Leaving Cert Higher level

Substitution

$$\sqrt{\frac{q^2+rp+r+4}{\frac{-q}{p}}}$$

For $p=3, q=-4$ and $r=7$

Sub in values for p, q and r

$$\sqrt{\frac{(-4)^2+(7)(3)+7+4}{\frac{-(-4)}{3}}}=6$$

Manipulate Formulae

Express a in terms of u, t and s

$$s=ut+\frac{1}{2}at^2$$

Get rid of brackets and fractions. Isolate letter of choice.

$$2s=2ut+at^2$$
$$2s-2ut=at^2$$
$$\frac{2s-2ut}{t^2}=a$$

For larger expansions we can use the Binomial Theorem

Binomial Theorem

$$(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

$(2x-y)^4$

$$\binom{4}{0}(2x)^4+\binom{4}{1}(2x)^3(-y)^1+\binom{4}{2}(2x)^2(-y)^2+\binom{4}{3}(2x)^1(-y)^3+\binom{4}{4}(-y)^4$$

$$16x^4-32x^3y+24x^2y^2-8xy^3+y^4$$

Simple Equation

$3(2x-1)=4x$

$$6x-3=4x$$
$$6x-4x=3$$
$$2x=3$$
$$x=\frac{3}{2}$$

Multiply to get rid of brackets, x's to one side, numbers to the other….

Forming Written Expression

The length of a rectangle is 5 times its width. The perimeter of the rectangle is 120m.

$width=x \qquad length=5x$

$$x+x+5x+5x=120$$
$$x=10$$

Multiplying Expressions

Opposite of factorising. Split the brackets and multiply each term in the 1st bracket by each term in the 2nd.

$(3x+2)(4x-3)$

$$=3x(4x-3)+2(4x-3)$$
$$=12x^2-9x+8x-6$$
$$=12x^2-x-6$$

Special Expansions (Learn)

$$(x+y)^2=x^2+2xy+y^2$$
$$(x-y)^2=x^2-2xy+y^2$$
$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$
$$(x-y)^3=x^3-3x^2y+3xy^2-y^3$$

Factorising – 4 types from Junior Cert

Highest Common Factor

$6x^2-15xy$

$3x(2x-5y)$

Difference of Squares

$4x^2-81$

$(2x+9)(2x-9)$

Quadratics

$6x^2-5x-21$

$(3x-7)(2x+3)$

Grouping

$ax+bx+ay+by$

$x(a+b)+y(a+b)$

$(x+y)(a+b)$

Special Factors

$$(x^2-y^2)=(x-y)(x+y)$$
$$(x^3-y^3)=(x-y)(x^2+xy+y^2)$$
$$(x^3+y^3)=(x+y)(x^2-xy+y^2)$$

Simplify by Factorising

$$\frac{x^2-xy}{x^2-y^2}$$
$$=\frac{x(x-y)}{(x+y)(x-y)}$$
$$=\frac{x}{x+y}$$

Dividing Fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\times\frac{d}{c}$$
$$=\frac{ad}{bc}$$

Simplify

$$\frac{x^3-27}{x^2-4}\div\frac{2x^2-6x}{x+2}$$
$$\frac{x^3-27}{x^2-4}\times\frac{x+2}{2x^2-6x}$$
$$=\frac{(x-3)(x^2+3x+9)}{(x+2)(x-2)}\times\frac{x+2}{2x(x-3)}$$
$$=\frac{x^2+3x+9}{2x(x-2)}$$

Algebraic Fractions

$$\frac{2}{3x-4}-\frac{1}{2x+1}=\frac{1}{2}$$

Find a common denominator which we can drop if there is an equals.

$$\frac{2(2)(2x+1)-1(2)(3x-4)=1(3x-4)(2x+1)}{2(3x-4)(2x+1)}$$
$$8x+4-6x+8=6x^2-8x+3x-4$$
$$0=6x^2-8x-8x+6x+3x-8-4-4$$
$$6x^2-7x-16=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(6)(-16)}}{2(6)}$$
$$x=\frac{7\pm\sqrt{433}}{12}$$
$$x=2.3\ \&-1.2$$

Show that $\frac{3x-5}{x-2}+\frac{1}{2-x}$ simplifies to a constant

$$\frac{3x-5}{x-2}+\frac{1}{2-x}=\frac{3x-5}{x-2}-\frac{1}{x-2}$$
$$=\frac{3x-5-1}{x-2}$$
$$=\frac{3x-6}{x-2}$$
$$=\frac{3(x-2)}{x-2}=3$$

Quadratic Graph by Completing the Square

Express $f(x)=x^2+10x+32$ in the form $(x+a)^2+b$

$$f(x)=x^2+10x+32$$
$$f(x)=x^2+10x+\left(\frac{10}{2}\right)^2+32-\left(\frac{10}{2}\right)^2$$
$$f(x)=x^2+10x+25+32-25$$
$$f(x)=x^2+10x+25+7$$
$$f(x)=(x+5)^2+7$$

Min point of curve $(-5,7)$ and axis of symmetry $x=-5$

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Surds

Properties of Surds:

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$\sqrt{a}\sqrt{a} = a$

Laws of Indices

$a^m . a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

$a^{-n} = \frac{1}{a^n}$

$a^{\frac{1}{n}} = \sqrt[n]{a}$ | $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

$a^0 = 1$

Laws of Logs

$\log_a m + \log_a n = \log_a mn$

$\log_a m - \log_a n = \log_a \frac{m}{n}$

$n \log_a m = \log_a m^n$

We can convert logs to any base using the following rule:

$\log_n m = \frac{\log_a m}{\log_a n}$

Log Equations

$\mathbf{2\log_3 x - \log_3(18 - x) = 1}$

$\log_3 x^2 - \log_3(18 - x) = 1$

$\log_3 \frac{x^2}{(18 - x)} = 1$

$3^1 = \frac{x^2}{18 - x}$

$54 - 3x = x^2$

$x^2 + 3x - 54 = 0$

$(x + 9)(x - 3) = 0$

$x = -9 \qquad x = 3$

Irrational Equations

$\mathbf{x - \sqrt{2x - 4} = 2}$

Isolate the surd. Square both sides and solve. May have to repeat. Always test solution.

$x - 2 = \sqrt{2x - 4}$

$(x - 2)^2 = \sqrt{2x - 4}^2$

$x^2 - 4x + 4 = 2x - 4$

$x^2 - 6x + 8 = 0$

$(x - 4)(x - 2) = 0$

$x = 4 \qquad x = 2$

$4 - \sqrt{2(4) - 4} = 2$
True

$2 - \sqrt{2(2) - 4} = 2$
True

$\mathbf{(x + \sqrt{x})(x - \sqrt{x}) = 6}$

$x(x - \sqrt{x}) + \sqrt{x}(x - \sqrt{x}) = 6$

$x^2 - x\sqrt{x} + x\sqrt{x} - \sqrt{x}^2 = 6$

$x^2 - x = 6$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3 \qquad x = -2$

Find the real number a such that for all $x \neq 9$

$\mathbf{\frac{x - 9}{\sqrt{x} - 3} = \sqrt{x} + a}$

$x - 9 = (\sqrt{x} + a)(\sqrt{x} - 3)$

$x - 9 = x - 3\sqrt{x} + a\sqrt{x} - 3a$

$3a - a\sqrt{x} = x - 3\sqrt{x} - x + 9$

$3a - a\sqrt{x} = 9 - 3\sqrt{x}$

$a(3 - \sqrt{x}) = 3(3 - \sqrt{x})$

$a = 3$

Rationalise Denominator

Express $\frac{1-\sqrt{3}}{1+\sqrt{3}}$ in the form $a\sqrt{3} - b$

Multiply above and below by the conjugate of the denominator to remove surds from denominator.

$\frac{1 - \sqrt{3}}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$

$= \frac{1(1 - \sqrt{3}) - \sqrt{3}(1 - \sqrt{3})}{1(1 - \sqrt{3}) + \sqrt{3}(1 - \sqrt{3})}$

$= \frac{1 - \sqrt{3} - \sqrt{3} + 3}{1 - \sqrt{3} + \sqrt{3} - 3}$

$= \frac{4 - 2\sqrt{3}}{-2}$

$= \sqrt{3} - 2$

Indices Equations

$\mathbf{\frac{8}{2^x} = 32}$

Try and get the same base number on the RHS and LHS. Equate the powers.

$8 = 32(2^x)$

$2^3 = 2^5(2^x)$

$2^3 = 2^{5+x}$

$3 = 5 + x$

$-2 = x$

Simplify $2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}}$

$2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} + 2^{\frac{1}{4}} = 4(2^{\frac{1}{4}})$

$= 2^2(2^{\frac{1}{4}})$

$= 2^{2\frac{1}{4}}$

Special Indices Example

$\mathbf{2^x - 6 + 2^{3-x} = 0}$

Let $y = 2^x$ and try make quadratic.

$2^x - 6 + 2^3 . 2^{-x} = 0$

$2^x - 6 + \frac{8}{2^x} = 0$

$y - 6 + \frac{8}{y} = 0$

$y^2 - 6y + 8 = 0$

$(y - 4)(y - 2) = 0$

$y = 4 \qquad y = 2$

$2^x = 4 \qquad 2^x = 2$

$x = 2 \qquad x = 1$

Logarithmic Expression

$p = \log_c x$, express $\log_c \sqrt{x} + \log_c cx$ in terms of p

$\log_c \sqrt{x} + \log_c cx$

$= \log_c x^{\frac{1}{2}} + \log_c c + \log_c x$

$= \frac{1}{2}\log_c x + \log_c c + \log_c x$

$= \frac{1}{2}p + 1 + p$

$= \frac{3}{2}p + 1$

We can also do these types of sum without introducing a y. Just looks trickier!

Solve $\frac{2}{e^x} = e^x - 1$

$2 = (e^x)^2 - e^x$

$(e^x)^2 - e^x - 2 = 0$

$(e^x - 2)(e^x + 1) = 0$

$e^x = 2 \qquad e^x = -1$

$x = \ln 2 \qquad$ No solution

General Term of a Binomial Expansion

Consider the binomial expansion of $\left(3x^2 + \frac{1}{2x}\right)^{10}$ in descending powers of x.
Find the coefficient of x^8

$t_{r+1} = \binom{n}{r} x^{n-r} y^r$

$t_{r+1} = \binom{10}{r} (3x^2)^{10-r} \left(\frac{1}{2x}\right)^r$

$= \binom{10}{r} (3x^{20-2r}) \left(\frac{1}{2x}\right)^r$

$= \binom{10}{r} (x^{20-2r})(3^{10-r}) \left(\frac{1}{2^r x^r}\right)$

$= \binom{10}{r} \left(\frac{3^{10-r}}{2^r}\right) (x^{20-3r})$

$20 - 3r = 8$

$3r = 12$

$r = 4$

$\binom{10}{r}\left(\frac{3^{10-r}}{2^r}\right)$

$= 210\left(\frac{3^6}{2^4}\right)$

$= 210\left(\frac{729}{16}\right)$

$= \frac{76545}{8}$

Identities/ Unknown Coefficients

$p(x + q)^2 + r = 2x^2 + 12x + 13$

for all x, find the value of p, of q and of r.

Remove all fractions or brackets. Equate like terms on each side.

$p(x + q)^2 + r = 2x^2 + 12x + 13$

$p(x^2 + 2qx + q^2) + r = 2x^2 + 12x + 13$

$px^2 + 2pqx + pq^2 + r = 2x^2 + 12x + 13$

$p = 2$

$2pq = 12$

$2(2)q = 12$

$4q = 12$

$q = 3$

$pq^2 + r = 13$

$(2)(3)^2 + r = 13$

$18 + r = 13$

$r = -5$

Solving Quadratic Equations

$\mathbf{2x^2 - 4x - 6 = 0}$

Factorise if possible, if not:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-6)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 + 48}}{4}$$

$$x = \frac{4 \pm \sqrt{64}}{4}$$

$$x = \frac{4 \pm 8}{4}$$

$x = \frac{4+8}{4}$ & $\frac{4-8}{4}$

$x = 3$ & -1

$2x^2 - 4x - 6 = 0$

$(2x + 2)(x - 3) = 0$

$2x + 2 = 0 \qquad x - 3 = 0$

$2x = -2 \qquad x = 3$

$x = -1$

Nature of Roots (Learn)

Real	$b^2 - 4ac \geq 0$
Equal	$b^2 - 4ac = 0$
No Real Roots	$b^2 - 4ac < 0$

$(k-1)x^2 - 6x + (k-1)$ has equal roots, find k.

$b^2 - 4ac = 0$

$(-6)^2 - 4(k-1)(k-1) = 0$

$36 - 4(k^2 - 2k + 1) = 0$

$36 - 4k^2 + 8k - 4 = 0$

$-4k^2 + 8k + 32 = 0$

$k^2 - 2k - 8 = 0$

$(k-4)(k+2) = 0$

$k = 4$ and $k = -2$

Forming a Quadratic Equation

$x^2 - (sum\ of\ the\ roots)x + (product\ of\ the\ roots) = 0$

Sum and Product of the Roots of a Quadratic

The quadratic equation $x^2 + bx + c = 0$ can be written

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

If α and β are the roots of $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then:

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Useful α and β identity

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

More relevant on 'old course'.

Solving Cubic Equations

Find the 3 integer roots of
$f(x) = x^3 - 2x^2 - 5x + 6$

We must guess 1st root (if not given) by subbing in for x. It will be factor of 6. Start with $x = 1$, the try $x = -1$ and so on.

$f(1) = (1)^3 - 2(1)^2 - 5(1) + 6$

$= 1 - 2 - 5 + 6 = 0$

If $x = -1$ is a root then $(x - 1)$ is a factor.

```
          x² − x − 6
x − 1 | x³ − 2x² − 5x + 6
        x³ − x²
        ---------
           −x² − 5x + 6
           −x² + x
           ------------
                −6x + 6
                −6x + 6
                -------
                      0
```

$(x - 1)(x^2 - x - 6) = 0$

$(x - 1)(x - 3)(x + 2) = 0$ Factors

$x = 1 \quad x = 3 \quad x = -2$ Roots

One root of the equation $px^2 + qx + r = 0$ is four times the other.
Show that $4q^2 - 25pr = 0$

Let the roots be α and 4α

$px^2 + qx + r = 0$

$x^2 + \frac{q}{p}x + \frac{r}{p} = 0$

Sum of the roots

$\alpha + 4\alpha = -\frac{q}{p}$

$5\alpha = -\frac{q}{p}$

$\alpha = -\frac{q}{5p}$

Product of roots

$(\alpha)(4\alpha) = \frac{r}{p}$

$4\alpha^2 = \frac{r}{p}$

$4(-\frac{q}{5p})^2 = \frac{r}{p}$

$\frac{4q^2}{25p^2} = \frac{r}{p}$

$4pq^2 = 25p^2r$

$4q^2 - 25pr = 0$

Using Factor Theorem

$x - 3$ and $x + 2$ are factors of $f(x) = 3x^3 + mx^2 - 17x + n$.
Find m and n

If $x - 3$ is a factor then $x = 3$ is a root and we can sub this into equation.

$f(3) = 3(3)^3 + m(3)^2 - 17(3) + n = 0$

$81 + 9m - 51 + n = 0$

$9m + n = -30$

$f(-2) = 3(-2)^3 + m(-2)^2 - 17(-2) + n = 0$

$-24 + 4m + 34 + n = 0$

$4m + n = -10$

Solve simultaneous equation

$9m + n = -30$

$4m + n = -10$

$5m = -20$

$m = -4$

$4m + n = -10$

$4(-4) + n = -10$

$-16 + n = -10$

$n = 6$

$m = -4$ & $n = 6$

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Quadratic Factor of a Cubic Function

Given that $x^2 - ax - 3$ is a factor of $x^3 - 5x^2 + bx + 9$ where $a, b \in R$ find the value of a and b.

Let $x + k$ be the other factor. If it was $\mathbf{2x^3 - 5x^2} + \cdots$ we'd use $(2x + k)$

Therefore

$(x + k)(x^2 - ax - 3) = x^3 - 5x^2 + bx + 9$

$x^3 - ax^2 - 3x + kx^2 - akx - 3k = x^3 - 5x^2 + bx + 9$

$x^3 + (-a + k)x^2 + (-3 - ak)x - 3k = x^3 - 5x^2 + bx + 9$

Equating the coefficients of like terms:

$-a + k = -5$	$-3 - ak = b$	$-3k = 9$
$-a + (-3) = -5$	$-3 - (2)(-3) = b$	$k = -3$
$-a - 3 = -5$	$-3 + 6 = b$	
$\mathbf{2 = a}$	$\mathbf{3 = b}$	

Inequalities

$\mathbf{5x + 1 \leq 4x + 3 \quad , x \in R}$

$5x - 4x \leq 3 - 1$
$x \leq 2$

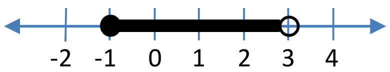
Treat this like an equation with x's to one side and numbers to the other. $x \in N, x \in Z$ use dots. $x \in R$ use shading.

Compound Inequalities

$\mathbf{-2 \leq 5x + 3 < 18 \quad , x \in R}$

Split into two inequalities and solve as before.

$-2 \leq 5x + 3$	$5x + 3 < 18$
$-2 - 3 \leq 5x$	$5x < 18 - 3$
$-5 \leq 5x$	$5x < 15$
$-1 \leq x$	$x < 3$



Abstract Inequalities

Show that $a^2 + b^2 \geq 2ab$

Any $(real\ number)^2 \geq 0$

Bring to one side and factorise to make something squared.

$a^2 - 2ab + b^2 \geq 0$
$(a - b)(a - b) \geq 0$
$(a - b)^2 \geq 0$

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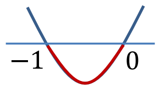
Rational Inequalities

$$\frac{3x + 1}{x + 1} \leq 1$$

Cannot cross multiply as $x + 1$ may be negative. Multiply each side by $(x + 1)^2$. Solve the quadratic.

$\frac{3x + 1}{x + 1}(x + 1)^2 \leq 1(x + 1)^2$
$(3x + 1)(x + 1) \leq 1(x + 1)^2$
$3x^2 + 3x + x + 1 \leq x^2 + 2x + 1$
$2x^2 + 2x \leq 0$
$2x(x + 1) = 0$
$x = 0 \qquad x + 1 = 0$
$\qquad\qquad x = -1$

$-1 < x \leq 0$



Note: $x \neq -1$

Modulus/ Absolute Value

$\mathbf{|x - 2| = 3}$

Isolate the modulus and then square both sides.

$x^2 - 4x + 4 = 9$
$x^2 - 4x - 5 = 0$
$(x - 5)(x + 1) = 0$
$x = 5 \qquad x = -1$

If $|x| = 4$
then $x = 4$
or $x = -4$

Modulus Inequality

$\mathbf{|4x + 7| > 1}$

$|4x + 7|^2 > 1^2$
$16x^2 + 56x + 49 > 1$
$16x^2 + 56x + 48 > 0$
$2x^2 + 7x + 6 > 0$
$(2x + 3)(x + 2) = 0$
$2x + 3 = 0 \quad x + 2 = 0$
$2x = -3 \qquad x = -2$

$x = -\frac{3}{2}$

$x > -\frac{3}{2} \qquad x < -2$

Simultaneous Equations – 2 unknowns (linear)

Solve $\quad \mathbf{4x + 16y = 20}$
$\qquad\qquad \mathbf{2x - 3y = -1}$

Multiply one or both lines to make coefficients of one of the variables the same. Cancel down and solve.

$4x + 16y = 20$
$2x - 3y = -1 \quad \times -2$

$4x + 16y = 20$
$-4x + 6y = 2$
$22y = 22$
$\mathbf{y = 1}$

$4x + 16y = 20$
$4x + 16(1) = 20$
$4x + 16 = 20$
$4x = 4$
$\mathbf{x = 1}$

Simultaneous Equations – 2 unknowns (linear & non-linear)

Solve $\quad \mathbf{2x + y = 10}$
$\qquad\qquad \mathbf{x^2 + y^2 - 4x - 2y = 0}$

Take the linear expression and express one variable in terms of the other. Sub this into the non-linear and solve.

$2x + y = 10$
$y = 10 - 2x$

$x^2 + y^2 - 4x - 2y = 0$
$x^2 + (10 - 2x)^2 - 4x - 2(10 - 2x) = 0$
$x^2 + 100 - 40x + 4x^2 - 4x - 20 + 4x = 0$
$5x^2 - 40x + 80 = 0$
$x^2 - 8x + 16 = 0$
$(x - 4)(x - 4) = 0$
$\mathbf{x = 4}$

$y = 10 - 2x$
$y = 10 - 2(4)$
$\mathbf{y = 2}$

Simultaneous Equations – 3 unknowns

$\mathbf{x + y + z = 16} \quad ①$
$\mathbf{\frac{5}{2}x + y + 10z = 40} \quad ②$
$\mathbf{2x + \frac{1}{2}y + 4z = 21} \quad ③$

① $x + y + z = 16 \quad \times -1$
② $\frac{5}{2}x + y + 10z = 40$

$-x - y - z = -16$
$\frac{5}{2}x + y + 10z = 40$
$\frac{3}{2}x + 9z = 24 \quad$ A

② $\frac{5}{2}x + y + 10z = 40$
③ $2x + \frac{1}{2}y + 4z = 21 \quad \times -2$

$\frac{5}{2}x + y + 10z = 40$
$-4x - y - 8z = -42$
$-\frac{3}{2}x + 2z = -2 \quad$ B

A $\quad \frac{3}{2}x + 9z = 24$
B $\quad -\frac{3}{2}x + 2z = -2$
$11z = 22$
$\mathbf{z = 2}$

A $\quad \frac{3}{2}x + 9z = 24$
$\frac{3}{2}x + 9(2) = 24$
$\frac{3}{2}x + 18 = 24$
$\frac{3}{2}x = 6$

$\mathbf{x = 4}$

① $\quad x + y + z = 16$
$4 + y + 2 = 16$
$y = 16 - 4 - 2$
$\mathbf{y = 10}$